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We consider the evolution of scalar perturbations in a class of non-singular bouncing universes obtained with higher-order corrections to the low-energy bosonic string action. We show that previous studies have relied on a singular evolution equation for the perturbations. From a simple criterium we show that scalar perturbations cannot be described at all times by an homogeneous second-order perturbation equation in pre-big bang type universes if we are to regularise the background evolution with higher-order curvature and string coupling corrections, and we propose a new system of first-order coupled differential equations. Given a bouncing cosmological background with inflation driven by the kinetic energy of the dilaton field, we obtain numerically the final power spectra generated from the vacuum quantum fluctuations of the metric and the dilaton field during inflation. Our result shows that both Bardeen's potential,  $\Phi(\eta, k)$ , and the curvature perturbation in the uniform curvature gauge,  $\mathcal{R}(\eta, k)$ , lead to a blue spectral distribution long after the transition.

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## I. INTRODUCTION

Although current observations [1–3] of anisotropies in the cosmic microwave background radiation (CMBR) strongly support the paradigm of scalar field potential-driven inflation as being the source of primordial density fluctuations, leading to an almost flat power spectrum [4, 5], this cosmological model is not short of conceptual problems [6]. One such loophole is the initial singularity which persists in standard inflation [7], but may eventually be addressed within a more fundamental theory of quantum gravity. String theory for instance admits a class of cosmological models where a period of super-inflation is driven by the kinetic energy of a scalar field. This “pre-big bang” phase with growing coupling and growing curvature is expected to last until the background evolution reaches a regime of maximal curvature, the bounce, where pure string effects may eventually turn the evolution to an expanding Friedmann-Lemaître (FL) regime. To represent a viable alternative to standard inflation, these cosmological models should also reproduce the high-accuracy measurements of the CMBR. To date, the late time spectral distribution of adiabatic density perturbations in pre-big bang type universes remains the subject of an intense debate. Some authors [8–11] recently argued that a scale invariant spectrum (for the ekpyrotic model [9, 10]) or a red spectrum (for the pre-big bang scenario [12, 13]) might emerge from the bouncing regime, while others [14–16] favoured a steep blue spectrum with  $n \simeq 3$  (or  $n = 4$ , respectively). It is thus of prime interest to determine precisely the spectral distribution of cosmological models which transit from a collapsing phase to an expanding FL regime. Since this discrepancy in the final spectral distribution results from applying different matching procedures to tree-level singular models, a regularised background evolution could in principle provide the framework to discriminate between these opposite claims.

One possibility of smoothing out a curvature singularity of the background evolution is to enhance the low-energy effective action of string theory with higher-order curvature corrections (e.g., see [17–20]). The general feature that emerges from this scheme is that higher-order corrections generally saturate the growth of spacetime curvature, while quantum loop corrections are required to violate the null energy condition and trigger a graceful exit to the post-bounce decelerating phase. The inclusion of higher-order corrections in the perturbation equations eventually modify the evolution of adiabatic density perturbations [21]. Lately, these perturbation equations have been used [22] to study the ekpyrotic model. In this work, the authors have followed the perturbations through a regularised transition and have obtained a final  $n \simeq 3$  spectrum. Here we shall argue that their method relied on a singular perturbation equation. We will then propose a new homogeneous system of coupled linear differential equations involving Bardeen's potential  $\Phi$  and curvature perturbation in the comoving gauge,  $\mathcal{R}$ . This system remains well-defined at all times. Numerical simulations then provide us with post-bounce spectral indices which we find are identical,  $n_\Phi = n_{\mathcal{R}} \simeq 3$  (or  $n_\Phi = n_{\mathcal{R}} = 4$  for the pre-big bang scenario). Thus our result appears in contradiction with the spectral index

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advocated by Durrer and Vernizzi [8] and favours the steep blue spectrum put forward in [23] and others.

The paper is organised as follows. In Sec. II we introduce a general action including possible curvature corrections up to fourth order in derivatives and we derive the corresponding field equations. Section III is devoted to studying the classical evolution of adiabatic density perturbations. We first comment on previous results and demonstrate that, in our particular class of cosmologies, the perturbation equations cannot be reduced to a set of one decoupled homogeneous second order differential equation for each variable. Given a background evolution, we then confront our predictions for the late time spectral distributions by numerically integrating a new set of perturbations equations. Finally, in Sec. IV we recall our main results and discuss possible extensions. For the sake of clarity, we leave the details of the calculation to the Appendix.

## II. BACKGROUND EVOLUTION

As a starting point, we consider the following  $D$ -dimensional effective action

$$\mathcal{S} = \frac{1}{\ell^{D-2}} \int d^D x \sqrt{-g} \left[ \frac{1}{2} \mathcal{F}(\phi) R - \frac{1}{2} \omega(\phi) \phi^{;\mu} \phi_{;\mu} - \mathcal{V}(\phi) + \frac{1}{2} \alpha' \lambda \mathcal{L}^{(c)} \right], \quad (1)$$

with  $\ell$  a spacetime-length parameter and the higher-order correction terms are given by

$$\mathcal{L}^{(c)} \equiv \xi(\phi) [c_1 R_{GB}^2 + c_2 G^{\mu\nu} \phi_{;\mu} \phi_{;\nu} + c_3 \square \phi \phi^{;\mu} \phi_{;\mu} + c_4 (\phi^{;\mu} \phi_{;\mu})^2], \quad (2)$$

where  $R_{GB}^2 = R_{\mu\nu\lambda\rho}^2 - 4R_{\mu\nu}^2 + R^2$  is the Gauss-Bonnet combination. Here  $\mathcal{F}(\phi)$ ,  $\omega(\phi)$  and  $\xi(\phi)$  are algebraic functions of a dimensionless scalar field  $\phi$ , and with the potential  $\mathcal{V}(\phi)$  we leave open the possibility of scalar field self-interactions. Through the lagrangian density  $\mathcal{L}^{(c)}$ , we allow for the inclusion of terms with higher numbers of derivatives such as contracted quadratic products of the curvature tensor and define its associated energy-momentum tensor by  $\frac{1}{2}\sqrt{-g} T^{(c)\mu\nu} \delta g_{\mu\nu} \equiv \delta(\sqrt{-g} \mathcal{L}^{(c)})$ . Extremising the general effective action Eq. (1) yields the covariant Euler-Lagrange equations for the cosmological model,

$$G_\nu^\mu = \frac{1}{2\mathcal{F}} \left[ T^{(0)\mu}_\nu + \alpha' \lambda T^{(c)\mu}_\nu \right], \quad (3)$$

$$2\omega \square \phi + \mathcal{F}_{,\phi} R + \omega_{,\phi} \phi^{;\mu} \phi_{;\mu} - 2\mathcal{V}_{,\phi} + \alpha' \lambda \Delta_\phi^{(c)} = 0, \quad (4)$$

where  $T^{(0)\mu}_\nu$  is the energy-momentum tensor of the tree-level terms. The variation of the lagrangian density Eq. (2) with respect to the scalar field  $\phi$  yields the  $\alpha'$  correction  $\Delta_\phi^{(c)}$  which satisfies  $T^{(c)\mu}_{\nu;\mu} = \Delta_\phi^{(c)} \phi_{;\nu}$  according to Bianchi's identity. Explicitly, they are given by [21]

$$T^{(0)\mu}_\nu \equiv 2(\omega + \mathcal{F}_{,\phi\phi}) \phi^{;\mu} \phi_{;\nu} + 2\mathcal{F}_{,\phi} \phi^{;\mu}{}_\nu - \delta_\nu^\mu [(\omega + 2\mathcal{F}_{,\phi\phi}) \phi^{;\sigma} \phi_{;\sigma} + 2\mathcal{F}_{,\phi} \square \phi + 2\mathcal{V}(\phi)], \quad (5)$$

$$\begin{aligned} T^{(c)\mu}_\nu &\equiv -8c_1 [(R^\mu_{\sigma\nu\tau} + R_{\nu\sigma} \delta_\tau^\mu - \delta_\nu^\mu R_{\sigma\tau}) \xi^{;\sigma\tau} + G^{\mu\sigma} \xi_{;\nu\sigma} - G_\nu^\mu \square \xi] + 2c_1 \xi \aleph_\nu^\mu \\ &+ c_2 \left\{ 2\xi [(\delta_\nu^\mu R_{\sigma\tau} - R^\mu_{\sigma\nu\tau}) \phi^{;\sigma} \phi^{;\tau} - R_\sigma^\mu \phi_{;\nu} \phi^{;\sigma} - R_\nu^\sigma \phi^{;\mu} \phi_{;\sigma}] \right. \\ &+ \xi [G_\nu^\mu \phi^{;\sigma} \phi_{;\sigma} + R \phi^{;\mu} \phi_{;\nu}] + (\square \xi \phi^{;\sigma} \phi_{;\sigma} - \xi^{;\sigma\tau} \phi_{;\sigma} \phi_{;\tau}) \delta_\nu^\mu \\ &+ \xi^{;\mu\sigma} \phi_{;\nu} \phi_{;\sigma} + \xi_{;\nu\sigma} \phi^{;\mu} \phi^{;\sigma} - \xi^{;\mu}{}_\nu \phi^{;\sigma} \phi_{;\sigma} - \square \xi \phi^{;\mu} \phi_{;\nu} \\ &+ \square \phi (\xi^{;\mu} \phi_{;\nu} + \xi_{;\nu} \phi^{;\mu}) + 2\delta_\nu^\mu (\xi^{;\sigma} \phi^{;\tau} \phi_{\sigma\tau} - \square \phi \xi^{;\sigma} \phi_{;\sigma}) + 2\xi^{;\sigma} \phi_{;\sigma} \phi^{;\mu}{}_\nu \\ &- [\xi_{;\sigma} \phi^{;\mu\sigma} \phi_{;\nu} + \xi^{;\sigma} \phi^{;\mu} \phi_{;\nu\sigma} + \xi^{;\mu} \phi_{;\nu\sigma} \phi^{;\sigma} + \xi_{;\nu} \phi^{;\mu\sigma} \phi_{;\sigma}] \\ &+ \xi [2\square \phi \phi^{;\mu}{}_\nu - \delta_\nu^\mu (\square \phi)^2 + \delta_\nu^\mu \phi^{;\sigma\tau} \phi_{;\sigma\tau} - 2\phi^{;\mu\sigma} \phi_{;\nu\sigma}] \Big\} \\ &+ c_3 \left\{ 2\xi [\phi^{;\mu\sigma} \phi_{;\nu} \phi_{;\sigma} + \phi^{;\mu} \phi_{;\nu} \phi^{;\sigma} \phi_{;\sigma} - \delta_\nu^\mu \phi^{;\sigma\tau} \phi_{;\sigma} \phi_{;\tau} - \square \phi \phi^{;\mu} \phi_{;\nu}] \right. \\ &+ \phi^{;\sigma} \phi_{;\sigma} [\xi^{;\mu} \phi_{;\nu} + \xi_{;\nu} \phi^{;\mu} - \delta_\nu^\mu \xi^{;\tau} \phi_{;\tau}] \Big\} \\ &+ c_4 \xi \phi^{;\sigma} \phi_{;\sigma} [\delta_\nu^\mu \phi^{;\tau} \phi_{;\tau} - 4\phi^{;\mu} \phi_{;\nu}], \end{aligned} \quad (6)$$

$$\aleph_\nu^\mu \equiv \frac{1}{2} \delta_\nu^\mu R_{GB}^2 + 4R^\mu_{\sigma\nu\tau} R^{\sigma\tau} - 2R^\mu_{\rho\sigma\tau} R_\nu^{\rho\sigma\tau} + 4R^{\mu\sigma} R_{\nu\sigma} - 2R R_\nu^\mu, \quad (7)$$

$$\begin{aligned}
\Delta_\phi^{(c)} \equiv & c_1 \xi_{,\phi} R_{GB}^2 + c_2 G^{\mu\nu} [\xi_{,\phi} \phi_{;\mu} \phi_{;\nu} - 2 \xi_{;\mu} \phi_{;\nu} - 2 \xi \phi_{;\mu\nu}] \\
& + c_3 [(\xi_{,\phi} \square \phi + \square \xi) \phi^{;\mu} \phi_{;\mu} - 2 \xi^{;\mu} \phi_{;\mu} \square \phi + 4 \xi^{;\mu} \phi^{;\nu} \phi_{;\mu\nu}] \\
& + 2 c_3 \xi [R^{\mu\nu} \phi_{;\mu} \phi_{;\nu} + \phi^{;\mu\nu} \phi_{;\mu\nu} - (\square \phi)^2] \\
& + c_4 [\xi_{,\phi} (\phi^{;\mu} \phi_{;\mu})^2 - 4 \phi^{;\mu} \phi_{;\mu} (\xi^{;\nu} \phi_{;\nu} + \xi \square \phi) - 8 \xi \phi^{;\mu} \phi^{;\nu} \phi_{;\mu\nu}] . 
\end{aligned} \tag{8}$$

Eqs. (3)–(8) form a complete set of covariant generalised Einstein equations for a background evolution including higher-order corrections in the metric and the dilaton field, hence extending their domain of validity to highly-curved regimes. For string solutions of physical interest, we wish to restrict to four physical spacetime dimensions, which we assume to be described by a FL metric with infinitesimal line element  $ds^2 = a^2(\eta) (-d\eta^2 + d\vec{x}^2)$ , where  $\eta$  denotes the conformal time and  $' \equiv d/d\eta$ . Inserting this line element in the above covariant equations yields a closed system of dynamical equations for the background evolution (cf. Appendix A 2). Since the quantity  $\mathfrak{N}_\nu^\mu$  encompasses a quadratic expression which vanishes in four spacetime dimensions on account of the algebraic identities satisfied by the Riemann tensor, we shall neglect it hereafter.

As an application of these equations, we may consider in particular the pre-big bang scenario of string cosmology [12, 13] which requires  $\mathcal{F} = e^{-\phi} = -\omega(\phi)$  and  $\mathcal{V}(\phi) = 0$  when considered at tree-level in the string frame. In string theory, the precise form of the higher-order corrections can be fixed by requiring that our effective action Eq. (1) reproduces the string theory  $S$ -matrix elements. At the next-to-leading order in the  $\alpha'$  expansion, this only constrains the coefficient of  $R_{\mu\nu\lambda\rho}^2$  with the result that the pre-factor for the Gauss-Bonnet term has to be  $c_1 = 1$ . But the lagrangian can still be shifted by field re-definitions which preserve the on-shell amplitudes, leaving the three remaining coefficients satisfying

$$\xi(\phi) = e^{-\phi}, \quad c_1 = 1, \quad c_3 = -\frac{1}{2} [c_2 + 2(c_1 + c_4)]. \tag{9}$$

The parameter  $\lambda$  allows us to move between different string theories :  $\lambda = 1/4, 1/8$  for the bosonic and heterotic string respectively, whereas for type II superstrings  $\lambda = 0$  and corrections start at higher order [24, 25]. Here we shall use  $\lambda = 1/4$  to agree with previous studies [26]. The natural setting  $c_2 = c_3 = 0$  leads to the well-known form which has given rise to most of the studies on corrections to the low-energy picture. In [18, 26], the authors demonstrated that this set of minimal tree-level  $\alpha'$  corrections regularises the singular behaviour of the low-energy pre-big bang scenario. Emerging from the asymptotic past vacuum along the low-energy exact pre-big bang solution, the (+) branch  $\phi' = (3 + \sqrt{3}) \mathcal{H}$  with increasing coupling and curvature, the  $\alpha'$  corrections drive the evolution to a fixed point of bounded curvature with a linearly growing dilaton in cosmic time (the  $\bullet$  in the left panel of Fig. 1). This suggests that quantum loop corrections — known to allow a violation of the null energy condition,  $p + \rho \geq 0$  — would permit the crossing of the Einstein bounce  $\mathcal{H}_e = \mathcal{H} - \phi'/2$  [27] (the subscript -e- denoting a quantity evaluated in the Einstein frame) and a graceful exit to a FL decelerated expansion in the post-big bang era, represented by the (-) branch with  $\phi' = (3 - \sqrt{3}) \mathcal{H}$ . Little is known about the exact form of quantum loop corrections and we take the freedom to parameterise them with an expansion in the string coupling  $g_s^2 = e^\phi$  of the form  $\xi(\phi) = e^{-\phi} + \xi_A + \xi_B e^\phi$  with constant free parameters  $\xi_A$  and  $\xi_B$ . An example of successful exit is shown in Fig. 1. In general, the combination of tree-level  $\alpha'$  and quantum loop corrections does not lead to a constant value for the dilaton in a finite amount of time. But this can be fixed by introducing a suitable potential or invoking particle production [18, 19].

Given that there exists a class of non-singular cosmologies based on these higher-order corrections, it is then natural to investigate the effect of the correction terms on the evolution of primordial scalar fluctuations.

### III. LINEAR PERTURBATIONS

#### A. The curvature perturbation $\mathcal{R}$

To study perturbations, we include small deviations of the background metric,  $g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$ , and for the scalar field,  $\phi$ . We know from [21] that an homogeneous second-order differential equation for the curvature perturbation on uniform energy density hypersurfaces,  $\mathcal{R}$ , can be derived from the general action Eq. (1) including higher-order corrections Eq. (2). To linear order, the decomposition in Fourier modes implies that each comoving wavenumber  $k$  evolves independently from other comoving modes according to

$$\mathcal{R}'' + 2 \frac{z'}{z} \mathcal{R}' + sk^2 \mathcal{R} = 0, \tag{10}$$

where we used the eigenstates of the Laplace-Beltrami operator,  $\Delta \mathcal{R} = -k^2 \mathcal{R}$ . Here, we introduced two functions of time,  $z = a\sqrt{Q}$  and  $s$ . For a minimally coupled massless field  $\phi$  and neglecting the contribution of the curvature

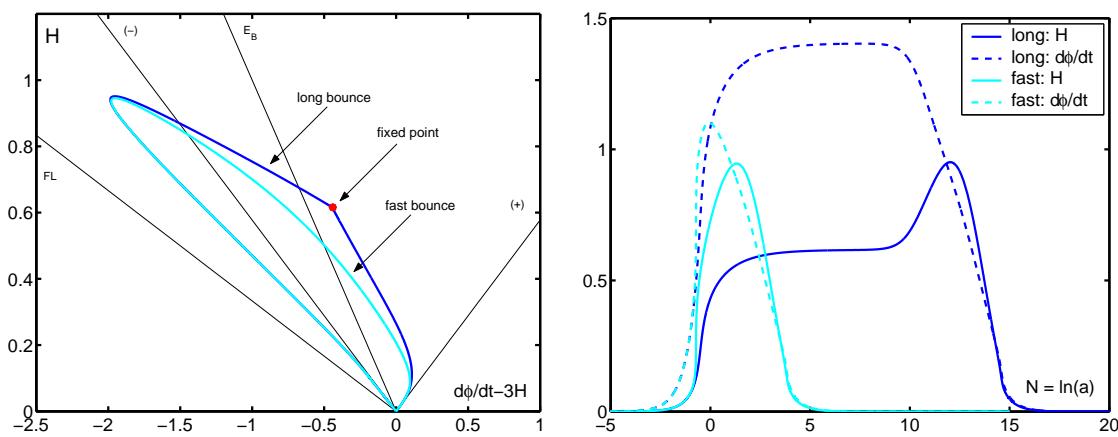


FIG. 1: We illustrate here two non-singular cosmological evolutions relying on the inclusion of higher-order curvature corrections. The figure on the left illustrates the flow diagram  $(\dot{\phi}, H)$ , where the (cosmic) time coordinate is just a parameter along the flow lines. The x-axis corresponds to  $\dot{\phi} = \dot{\phi} - 3H$  and the y-axis to  $H = \dot{a}/a$ , the Hubble expansion rate in cosmic time in the string frame. The  $(+)$ / $(-)$  branches correspond to the accelerated/decelerated phases of the pre-big bang scenario without potential. The line  $E_B$  represents a null Hubble parameter in the Einstein frame,  $H_e = 0$ . A typical non-singular evolution thus emerges from a weak-coupling and low-curvature state, located at the origin of the flow diagram. Then follows a dilaton-driven phase with increasing coupling and curvature along the  $(+)$  branch and runs into a perturbative singularity, unless higher-order corrections force the background to undergo a branch change,  $\dot{\phi} \geq 0 \rightarrow \dot{\phi} \leq 0$ , hence regulating the background evolution. Crossing over the Einstein bounce ensures that the Hubble parameter in the Einstein frame is henceforth positive. The figure on the right shows the corresponding evolution of  $H$  and  $\dot{\phi}$  as a function of the number of e-folds,  $N = \ln(a)$ . Both regular evolutions have been obtained for  $c_1 = -1$ ,  $c_2 = c_3 = 0$ ,  $c_4 = 1$ , with a loop control function  $\xi(\phi) = e^{-\phi} + \xi_A + \xi_B e^\phi$ . For the fast bounce, we set  $\xi_A = 1$  and  $\xi_B = -2.5 \cdot 10^{-3}$ . The initial conditions for the simulations have been set with respect to the lowest-order analytical solutions at  $\eta = -10^6$ . A long high curvature regime is easily obtained by trapping the background evolution in the  $\alpha'$ -asymptotic fixed point (denoted by a  $\bullet$ ), featuring a constant Hubble rate and linearly growing dilaton in cosmic time. This can be achieved by reducing the constant parameters  $\xi_A$  and  $\xi_B$ , or by shifting the initial value of the dilaton,  $\phi \rightarrow \phi - \phi_0$  with, e.g.,  $\phi_0 = 25$ . Eventually the string coupling  $g_s = e^{\phi/2}$  grows strong enough to allow for a violation of the null energy condition  $\rho + p \geq 0$  and enables a successful exit to the decelerated post-big bang regime.

corrections, these parameters reduce to  $Q \rightarrow (\phi'/\mathcal{H})^2$  and  $s \rightarrow +1$ . However, in the more general situation where the coupling remains unspecified and we allow for higher-order terms of the form of Eq. (2), we have (see [21] for an explicit derivation)

$$Q \equiv \frac{\omega\phi'^2 + 3\frac{(\mathcal{F}' + aQ_a)^2}{2\mathcal{F} + Q_b} + a^2Q_c}{\left[\mathcal{H} + \frac{\mathcal{F}' + aQ_a}{2\mathcal{F} + Q_b}\right]^2} = \frac{\phi'E_2}{a^2E_1^2} \Omega, \quad \Omega(\eta) \equiv E_2E_5 + 3E_1E_4, \quad (11)$$

$$s \equiv 1 + \frac{a^2Q_d + a\frac{\mathcal{F}' + aQ_a}{2\mathcal{F} + Q_b}Q_e + \left[\frac{\mathcal{F}' + aQ_a}{2\mathcal{F} + Q_b}\right]^2Q_f}{\omega\phi'^2 + 3\frac{(\mathcal{F}' + aQ_a)^2}{2\mathcal{F} + Q_b} + a^2Q_c}. \quad (12)$$

To be concise, the  $Q_i$ 's and  $E_i$ 's are complicated functions of the background time-dependent quantities, the  $Q_i$ 's representing the  $\alpha'$  modifications of the tree-level equation. Their explicit form is given in the Appendix. Introducing the canonical variable  $v \equiv z\mathcal{R}$  [28], the linearised wave equation (10) reduces to

$$v'' + [sk^2 - U(\eta)] v = 0, \quad U(\eta) = \frac{z''}{z}. \quad (13)$$

Through the external potential  $U(\eta)$ , the ‘‘pump’’ field  $z$  is responsible for the parametric amplification [29, 30] of the metric fluctuations. In the case where the background evolution undergoes a period of pole-like inflation (e.g., derived from induced gravity [31], scalar-tensor gravity [32], the (modified) pre-big-bang scenario [8, 12, 13] or the ekpyrotic model [9, 10, 33]), the pump field reduces to a power-law,  $z \propto a_e = |\eta|^q$ , hence  $U(\eta) = q(q-1)|\eta|^{-2}$ . Then Eq. (13) corresponds for  $s = 1$  to a Bessel equation and its general solution can be expressed as a superposition of

Hankel functions of the first and second kind which, in terms of our original variable, reads [21]

$$\mathcal{R} = \frac{\sqrt{\pi|\eta|}}{2z} \sum_{i=1}^2 d_i(k) H_\nu^{(i)}(k|\eta|) , \quad \nu \equiv \frac{1}{2} |1 - 2q| . \quad (14)$$

This solution satisfies the usual unitary condition between the coefficients  $|d_2|^2 - |d_1|^2 = 1$ , while choosing a pure positive frequency state in the asymptotic flat space time for  $\eta \rightarrow -\infty$  requires  $d_2 = 1$  and  $d_1 = 0$ .

In the large scale limit  $k/|\mathcal{H}| \rightarrow 0$ , Eq. (14) reduces to  $\mathcal{R} \simeq A(k) + B(k)|\eta|^{1-2q}$ , where the coefficients  $A$  and  $B$  are fixed by the exact solution (14). Defining the initial power spectrum by  $\mathcal{P}_x \equiv |x|^2 k^3 \propto k^{n_x-1}$ , the spectral index carried by each mode before the bounce is thus  $n_A = 3 + 2q$  and  $n_B = 5 - 2q$ . For the pre-big bang scenario with vanishing potential, we have  $q = 1/2$  and  $n_A = n_B = 4$ . However, adiabatic density fluctuations can also be described by the gauge-invariant Bardeen potential in the longitudinal gauge,  $\Phi_e = \Phi - \frac{1}{2}\delta\phi_\chi$ , which evolves according to  $\Phi_e'' + 6\mathcal{H}\Phi_e' + k^2\Phi_e = 0$  at tree-level in the Einstein frame. In the large scale limit,  $\Phi_e \simeq C(k)|\eta|^{-(1+2q)} + D(k)$  and consistent normalisation yields  $n_C = 1 - 2q = 0$  and  $n_D = 3 + 2q = 4$  for the pre-big bang scenario. Comparing the amplitudes of the modes  $C$  and  $D$ , we clearly have  $\Phi_{e,C} \gg \Phi_{e,D}$  right before the bounce,  $|\eta| \rightarrow 0$  (see [34] for the details). Naively, we therefore expect that the dominant spectral index after the bounce remains  $n_C$ . However, since in the radiation dominated era  $\Phi_e$  and  $\mathcal{R}$  differ only by a  $k$ -independent constant, general relativity requires them to have the same spectral distribution. Hence the claims that the spectral distribution of  $\mathcal{R}$  is unaltered during a bounce and that the dominant pre-bounce spectral index yields the late time post-bounce spectral distribution for  $\Phi_e$  are contradictory.

One possibility to discriminate among these incompatible results is to consider a regular background evolution. As discussed in [21], the corrections (2) genuinely alter the time-dependence of the variables  $z = a\sqrt{Q}$  and  $s$  during the high-curvature regime; the source  $Q$  of the effective potential for the perturbation may no longer have a monotonic growth and may even decrease at the onset or during a high-curvature regime. Apparently, there is also no restriction on the sign of the frequency shift occurring when the curvature corrections dominate the background dynamics:  $s$  may become negative or infinite depending on its particular form. Previous applications of Eqs. (10)–(12) to pre-big bang-like scenarios [21, 22] have however missed out the singular behaviour of  $Q$  during a graceful exit. Indeed the components of the  $Q$ -denominator  $E_1 = a^2(\mathcal{F}' + 2\mathcal{F}\mathcal{H} + aQ_g)$  have to cancel out each other at some particular time, since the asymptotic analytical branches of the pre-big bang scenario ( $\mathcal{F} = e^{-\phi}$  in the string frame) yields opposite signs for  $E_1$ :

$$-\infty < \eta < -\eta_s : \quad \phi' = (3 + \sqrt{3})\mathcal{H} \quad \rightarrow \quad E_1 = -(\sqrt{3} + 1)a^2\mathcal{H}e^{-\phi} < 0 , \quad (15)$$

$$\eta_s < \eta < +\infty : \quad \phi' = (3 - \sqrt{3})\mathcal{H} \quad \rightarrow \quad E_1 = (\sqrt{3} - 1)a^2\mathcal{H}e^{-\phi} > 0 , \quad (16)$$

where  $\eta_s$  represents the time at which higher-order corrections dominate the background dynamics. Furthermore, this is the case for any FL spatially-flat bouncing universe which is described by the action Eq. (1), as  $E_1 \propto \mathcal{H} - \phi'/2 = \mathcal{H}_e$  has to change sign during the transition between a collapsing phase and an expanding phase. Therefore the pump field  $z$  and the ratio  $z'/z$  are singular, which invalidates the use of Eq. (10) in the context of bouncing universes of the type considered here.

We now proceed to investigate yet another dynamical equation for the perturbation  $\mathcal{R}$ , for we are interested in obtaining the spectrum long after the transition from a regular equation.

## B. Coupled systems $(\Phi, \delta\phi_\chi)$ and $(\Phi, \mathcal{R})$

The approach we consider henceforth relies on the gauge-invariant Bardeen potentials,  $\Phi$  and  $\Psi$ , and the perturbation in the scalar field,  $\delta\phi_\chi$ . Here we shall present our main results, and leave the definition of these variables and an explicit derivation of the terms involved to Sec. A 3. At tree-level, it is well known [35] that these variables satisfy homogeneous second-order differential equations at all times. When higher-order corrections are included, however, the decoupled wave equation of each of these variables is plagued by a similar singular behaviour during the transition regime as the one of  $\mathcal{R} = -\Phi + \frac{\mathcal{H}}{\phi'}\delta\phi_\chi$ . To see this, we recall that the covariant equations (3)–(4) yield five redundant evolution equations in terms of the gauge-invariant perturbed variables  $\Phi$ ,  $\Psi$  and  $\delta\phi_\chi$ . They are the four components of the Einstein equation,  $(0)_0$ ,  $(0)_j$ ,  $(i)_i$  and the trace-free part  $( )_{tf} = (i)_j - \frac{1}{3}(k)_k\delta_j^i$ . Those are supplemented by the perturbed part of the scalar field equation  $(\phi)$ . The  $(0)_0$  and  $(0)_j$  equations are first order in derivatives of the perturbed variables, while the  $(i)_i$  and  $(\phi)$  are of second order. Of prime interest is the tracefree part which implies a linear background-dependent relation among the perturbation variables

$$\Psi = c_\Phi\Phi + c_{\delta\phi_\chi}\delta\phi_\chi , \quad (17)$$

where the coefficients  $c_\Phi$  and  $c_{\delta\phi_\chi}$  are functions of time (see Appendix),

$$c_\Phi = -E_3 E_2^{-1}, \quad \text{and} \quad c_{\delta\phi_\chi} = -E_6 E_2^{-1}. \quad (18)$$

In the limit  $\alpha' \rightarrow 0$ , we recover  $\Psi + \Phi = \delta\phi_\chi$  in the string frame, which corresponds to the usual relation  $\Psi_e + \Phi_e = 0$  when expressed in the Einstein frame. The relation among these variables is modified in a non-trivial way when the background dynamics is dominated by the curvature corrections, but the alteration does not exhibit any dependence with respect to the comoving wavenumber  $k$ . For a regularised background evolution with  $E_2 \neq 0$  at all times (this is the case for a wide range of models we have tested), Eq. (17) enables us to replace the variable  $\Psi = \Psi(\Phi, \delta\phi_\chi)$  in the perturbation equations. We can then manipulate the remaining equations to extract a set of decoupled dynamical equations for the perturbations  $\Phi$  and  $\delta\phi_\chi$ . However, these equations are plagued by a singular behaviour during the transition regime of the kind we previously discussed for  $\mathcal{R}$ . To demonstrate that we cannot obtain *regular* decoupled second-order wave equations for each of these variables, we focus on the  $\binom{0}{0}$  and  $\binom{0}{j}$  equations, which completely specify the dynamical evolution of the perturbations. These first-order equations can be written as  $\mathcal{A}X' + \mathcal{B}X = 0$  where  $X \equiv (\Phi, \delta\phi_\chi)^T$ , and  $\mathcal{A}(\eta)$  and  $\mathcal{B}(\eta, k)$  are  $2 \times 2$  matrices whose coefficients are complicated combinations of the background quantities. Provided that the matrix  $\mathcal{A}$  is invertible, i.e.,  $\det(\mathcal{A}) = -2\Omega \neq 0$  at all times, we can extract an homogeneous system of linear differential equations  $X' = CX$  with time-dependent coefficients  $C(\eta, k) = -\mathcal{A}^{-1}\mathcal{B}$ :

$$\begin{pmatrix} \Phi \\ \delta\phi_\chi \end{pmatrix}' = \begin{pmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} \\ \mathcal{C}_{21} & \mathcal{C}_{22} \end{pmatrix} \begin{pmatrix} \Phi \\ \delta\phi_\chi \end{pmatrix}. \quad (19)$$

This homogeneous linear system has a set of  $n = 2$  linearly independent solutions of the form

$$X(\eta, k) = T \exp \left( \int^{\eta} \mathcal{C}(\tilde{\eta}, k) d\tilde{\eta} \right) X(\eta_0, k), \quad (20)$$

where  $T$  is a time-ordering operator. If  $\mathcal{C}_{ij}(\eta)$  are continuous functions and initial conditions are prescribed, the system has a unique solution given by Eq. (20). Although every differential equation of order  $n$  can be rewritten as a first-order system of  $n$  equations, the system viewpoint is more general as we shall recall now. In principle, the system Eq. (19) can be reduced to a second-order differential equation by elimination of one of the variable, e.g.,  $\delta\phi_\chi$ . Indeed it is straightforward to obtain the homogeneous differential equation

$$\Phi'' - \left( \mathcal{C}_{11} + \mathcal{C}_{22} + \frac{\mathcal{C}'_{12}}{\mathcal{C}_{12}} \right) \Phi' + \left[ \mathcal{C}_{11}\mathcal{C}_{22} - \mathcal{C}_{12}\mathcal{C}_{21} + \frac{1}{\mathcal{C}_{12}} (\mathcal{C}_{11}\mathcal{C}'_{12} - \mathcal{C}'_{11}\mathcal{C}_{12}) \right] \Phi = 0. \quad (21)$$

But if  $\mathcal{C}_{11}$  or  $\mathcal{C}_{12}$  are not differentiable, or if  $\mathcal{C}_{12} = 0$  at some particular time  $\eta_*$ , the reduction to Eq. (21) will not be possible (leaving aside the trivial case for which  $\mathcal{C}_{12} = 0$  at all time).

For all background evolutions of the type discussed in Sec. II we have tested, we find that  $E_2 \neq 0$  and  $\Omega \neq 0$  during the whole evolution, while  $\mathcal{C}_{12}$  and  $\mathcal{C}_{21}$  change sign. Hence homogeneous second-order equations valid at all times for either  $\Phi$  or  $\delta\phi_\chi$  cannot be derived in our particular class of cosmologies, and we shall use Eq. (19) to determine the spectral distribution long after the high-curvature regime. Explicitly, the decomposition  $\mathcal{C}_{ij}(\eta, k) = \hat{\mathcal{C}}_{ij}(\eta) + k^2 \check{\mathcal{C}}_{ij}(\eta)$  yields

$$\hat{\mathcal{C}}(\eta) = \begin{pmatrix} \mathcal{H}c_\Phi & \mathcal{H}c_{\delta\phi} - \Xi \\ \phi'c_\Phi & \phi'c_{\delta\phi} + \Pi \end{pmatrix}, \quad \check{\mathcal{C}}(\eta) = \Omega^{-1} \begin{pmatrix} -E_2 E_4 & -E_4^2 \\ E_2^2 & E_2 E_4 \end{pmatrix}, \quad (22)$$

where the coefficients  $c_\Phi(\eta)$  and  $c_{\delta\phi}(\eta)$  are given in Eq. (18),  $\Omega(\eta)$  is given by Eq. (11) and we define

$$\Xi(\eta) \equiv (E_4 E_{10} + 2E_5 E_{11}) / (2\Omega), \quad (23)$$

$$\Pi(\eta) \equiv -[\Xi + (\mathcal{H}/\phi')'] \phi'/\mathcal{H}. \quad (24)$$

Hence, Eqs. (19) and (22) provide us with the necessary tools to determine the spectral distribution of  $\Phi$  and  $\delta\phi_\chi$  long after the high-curvature regime. If numerical simulations were close to infinite precision, we could then deduce the evolution of  $\mathcal{R} = -\Phi + \frac{\mathcal{H}}{\phi'} \delta\phi_\chi$  and its late time power spectrum. This in turn would enable us to discriminate among the incompatible claims resulting from different matching procedures used in the tree-level analysis. Unfortunately, lack of numerical precision inevitably spoils the spectral distribution of  $\mathcal{R}$ . In a string cosmology background, we recall that metric and field perturbations are normalised with respect to the maximal amplified frequency  $k_{max} \equiv \text{Max}(\mathcal{H})$ , for which the perturbation variables have similar amplitudes,  $|\delta_\Phi| \approx |\delta_{\delta\phi_\chi}| \approx |\delta_{\mathcal{R}}|$ . Therefore, as long as  $\Phi$  and  $\delta\phi_\chi$  carry red spectra, we will have  $|\delta_\Phi| \approx |\delta_{\delta\phi_\chi}| \approx (k_{max}/k)^2 |\delta_{\mathcal{R}}|$  since we expect  $\mathcal{R}$  to have a steep blue  $k^3$  spectrum. But

for sufficiently small wavenumbers,  $k \ll k_{max}$ , the spectrum of  $\mathcal{R}$  obtained from  $\mathcal{R} = -\Phi + \frac{\mathcal{H}}{\phi'} \delta\phi_\chi$  will be dominated by the imprecision in numerically evolving  $\Phi$  and  $\delta\phi_\chi$ ; the deduced spectrum of  $\mathcal{R}$  would thus be red, i.e., a measure of the inaccuracy of the numerical simulations. To avoid such contamination and erroneous conclusion, we may instead consider the variable  $Y \equiv (\Phi, \mathcal{R})$ , whose dynamics also satisfies an homogeneous system of linear differential equations of first order,  $Y' = \mathcal{D}(\eta, k)Y$ :

$$\begin{pmatrix} \Phi \\ \mathcal{R} \end{pmatrix}' = \begin{pmatrix} \mathcal{H}c_\Phi + \phi' c_{\delta\phi} - \frac{\phi'}{\mathcal{H}} \Xi & \phi' c_{\delta\phi} - \frac{\phi'}{\mathcal{H}} \Xi \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \Phi \\ \mathcal{R} \end{pmatrix} + \frac{k^2}{\Omega \mathcal{H} \phi'} \begin{pmatrix} -E_1 E_4 \phi' & -E_4^2 \phi'^2 \\ E_1^2 & E_1 E_4 \phi' \end{pmatrix} \begin{pmatrix} \Phi \\ \mathcal{R} \end{pmatrix}, \quad (25)$$

where we have decomposed again the matrix elements according to  $\mathcal{D}_{ij}(\eta, k) = \hat{\mathcal{D}}_{ij}(\eta) + k^2 \check{\mathcal{D}}_{ij}(\eta)$ . Eq. (25) is the main result of this analysis: It determines the evolution of the curvature perturbation  $\mathcal{R}$  on super-Hubble scales. Indeed, the background equations, including those higher-order corrections needed to regularise the background evolution, imply at all times the exact cancellation of some coefficients, such that  $\hat{\mathcal{D}}_{21} \equiv 0 \equiv \hat{\mathcal{D}}_{22}$ . Hence  $\mathcal{R}$  remains nearly constant on super-Hubble scales, its evolution entering only as a  $(k/\mathcal{H})^2$  correction, while Bardeen's potential may evolve drastically on super-Hubble scales. This clearly suggests that, if the perturbation variables  $\mathcal{R}$  and  $\Phi$  are to yield the same spectral index long after the transition, the pre-bounce growing mode of the Bardeen potential has to be fully converted into its decaying mode during the high-curvature regime. This is confirmed by noting that the leading source term of  $\Phi'$  on super-Hubble scales, i.e., the coefficient  $\hat{\mathcal{D}}_{11}$  goes from positive to negative during the high-curvature regime. The growth of the Bardeen potential is thus turned into a rapid decay during the high-curvature regime.

The complexity of the matrix elements of Eq. (25) does not enable us to solve this system of differential equations exactly, and we have to resort to numerical integration of the perturbation equations to confirm our prediction about the post-bounce spectral distributions. Initial conditions in the asymptotic past can be set according to

$$\begin{pmatrix} \Phi \\ \delta\phi_\chi \end{pmatrix} = \frac{\phi'}{E_1^2} \begin{pmatrix} -E_1 E_4 & k^{-2} \mathcal{H} \Omega \\ E_1 E_2 & k^{-2} \phi' \Omega \end{pmatrix} \begin{pmatrix} \mathcal{R} \\ \mathcal{R}' \end{pmatrix}, \quad (26)$$

where we may use the exact tree-level pre-big bang solution (14) for  $\mathcal{R}$  and its first time derivative. We recall that these relations cannot be used at all times since  $E_1$  is forced to change sign during the course of the background evolution, and we have also shown that there is no second-order homogeneous equation for  $\mathcal{R}$  valid at all time. However these relations are fully adequate to initialise the perturbations  $\Phi$  and  $\delta\phi_\chi$  at early times  $\eta \ll -\eta_s < 0$ .

Figure 2 illustrates the results of numerical integration based on the system Eq. (25) for the pre-big bang scenario of string cosmology. We observe that the pre-bounce dominant mode of the Bardeen potential carrying the red spectrum is fully converted into the decaying post-bounce mode. Indeed, as shown in [34], a kink in the spectral distribution arises only in the situation where the growing mode of the pre-bounce phase is fully converted into the decaying mode after the transition, and one has to wait a sufficiently long time for the decaying mode to decay and the final growing mode to dominate. As a result, the pre-bounce subdominant mode yields, long after the bounce, the spectral index relevant for the observed anisotropies in the cosmic microwave background; it corresponds to a steep blue spectrum with  $n_\Phi = 3 + 2q = 4$ . Hence, although the perturbation variables  $\Phi$  and  $\mathcal{R}$  carry different spectral distributions before the bounce, our numerical simulations confirm that they have the same spectral index sufficiently long after the transition when the decaying mode of  $\Phi$  has died away. This favours the argument put forward in [14–16], where the authors have shown that this is exactly what happens if a matching between the pre- and post-bounce epochs is defined by a vanishing jump in the metric and the second fundamental form on the constant energy hypersurface.

#### IV. CONCLUSION

Even though we have not been able to demonstrate from first principles that the pre-bounce growing mode of the Bardeen potential  $\Phi$  (which carries a red spectrum) has to be fully converted into the decaying mode during the high-curvature regime, we have derived a new master equation which enables us to describe the evolution of adiabatic density perturbations on super-Hubble scales. This master equation takes the form of an homogeneous system of linear differential equations of first order, explicitly coupling the Bardeen potential  $\Phi$  and the curvature perturbation  $\mathcal{R}$ . On the one hand, we have shown that such system cannot be reduced to second order homogeneous differential equations for the class of cosmologies considered here, hence correcting erroneous claims in the literature. On the other hand, this master equation clearly indicates that the curvature perturbation is nearly constant on super-Hubble scales, its time-dependence entering as a  $(k/\mathcal{H})^2$  correction. This implies that the pre-bounce growing mode of  $\Phi$  is fully converted into the decaying mode during the transition. This is confirmed by numerical simulations which indicate that the dominant mode of the Bardeen potential carries a blue spectrum long after the transition. As we do

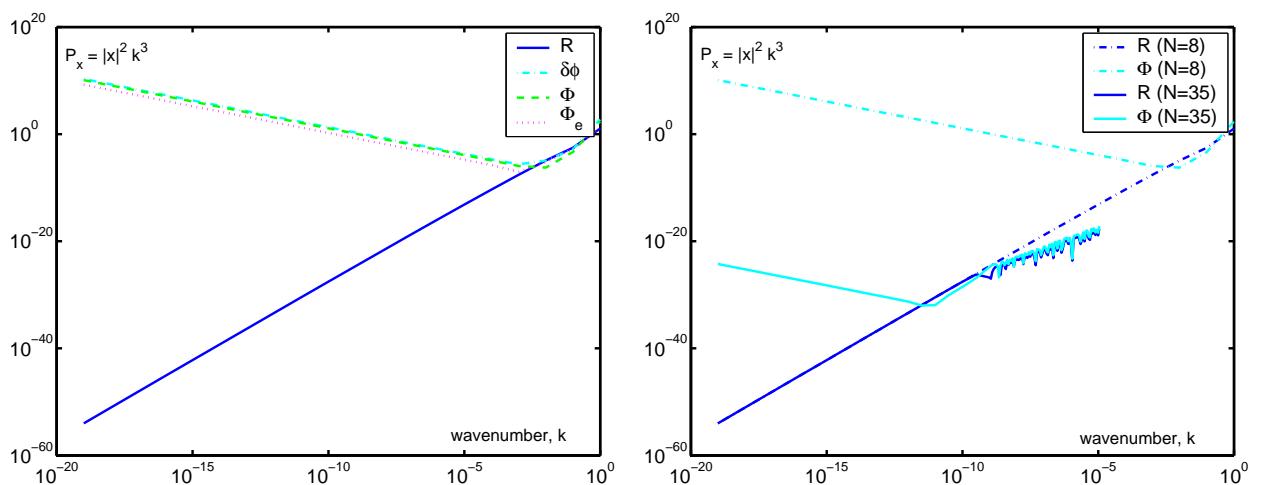


FIG. 2: We illustrate here the post-bounce spectral distributions of adiabatic density perturbations in the non-singular cosmological background of Fig. 1. We considered the high-curvature transition of short duration in order to reduce numerical uncertainty. The picture on the left shows  $\mathcal{P}_x = |x|^2 k^3$  for  $x \in \{\mathcal{R}, \delta\phi_x, \Phi, \Phi_e\}$  evaluated shortly after the bounce,  $N = \ln(a) \simeq 8$ . The wavenumber  $k$  is given in units of  $k_{max} \equiv \text{Max}(\mathcal{H})$ . The red spectrum attached to the decaying mode of  $\Phi$  is still dominant on nearly all scales, while  $\mathcal{R}$  carries a blue spectrum. On the panel on the right we compare the spectral distributions  $\mathcal{P}_x = |x|^2 k^3$  for  $x \in \{\mathcal{R}, \Phi\}$  evaluated at two different times after the bounce :  $N \simeq 8$  and  $N \simeq 35$ . The amplitude of  $\mathcal{P}_\Phi$  is decreasing in time until it reaches its sub-dominant constant mode, which carries a blue spectrum. Hence, both perturbation variables  $\Phi$  and  $\mathcal{R}$  lead to the same blue primordial power spectrum long after the transition. Finally, back inside the Hubble radius, the spectra start to oscillate and we have  $\Phi = \pm\sqrt{3}\mathcal{R} + \text{cst}$ .

not expect that a smooth transition from the decelerating post-big bang phase to the usual radiation-dominated FL epoch ( $\phi = \text{cst.}$ ) modifies the spectral distribution, we find that the final spectrum of adiabatic density perturbation is strongly blue tilted for the class of cosmologies considered here. This result is in complete agreement with [36] where the bounce is triggered by a non-local potential instead of higher-order curvature corrections.

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## APPENDIX A

In this appendix, we provide the necessary tools to reproduce the results discussed in the main text. After introducing our notation, we present the different components entering the background equations of motion. Then we derive the gauge-invariant expressions which contribute to the linear perturbation equations. Finally, we introduce our matrix notation, and give all the relevant terms in order to derive the systems of differential equations governing the evolution of adiabatic density perturbations.

### 1. Notation

To linear order the most general perturbation in a four-dimensional FL spacetime is given by the infinitesimal line element

$$ds^2 = -a^2(1+2\tau)d\eta^2 - 2a^2(\beta_{|i} + B_i)d\eta dx^i + a^2 \left[ g_{ij}^{(3)}(1+2\varphi) + 2\gamma_{|ij} + 2C_{(i|j)} + 2h_{ij} \right] dx^i dx^j .$$

Under the gauge transformation

$$\eta \mapsto \eta + \zeta^0(\eta, \mathbf{x}) , \quad x^i \mapsto x^i + g^{(3)ij} \zeta^{(s)}_{|j}(\eta, \mathbf{x}) + \zeta^{(v)i}(\eta, \mathbf{x}) , \quad (\text{A1})$$

the induced changes in the perturbed part of the metric read

$$\begin{aligned} \tau &\mapsto \tau - \mathcal{H}\zeta^0 - \zeta^{0\prime} , & \varphi &\mapsto \varphi - \mathcal{H}\zeta^0 , & \gamma &\mapsto \gamma - \zeta^{(s)} , & h_{ij} &\mapsto h_{ij} , \\ \beta &\mapsto \beta + \zeta^{(s)\prime} - \zeta^0 , & B_i &\mapsto B_i + \zeta_i^{(v)\prime} , & C_i &\mapsto C_i - \zeta_i^{(v)} , \end{aligned} \quad (\text{A2})$$

where we use the notation  $\prime \equiv d/d\eta$  and  $\mathcal{H} \equiv a'/a$ . The spatial gauge-invariance of the shear,  $\chi \equiv a(\beta + \gamma') \mapsto \chi - a\zeta^0$ , enables us to define an infinite number of gauge-invariant quantities. For instance, the Bardeen potentials are built exclusively from metric variables

$$\Phi \equiv \varphi - \frac{\mathcal{H}}{a}\chi , \quad \text{and} \quad \Psi \equiv \tau - \frac{\chi'}{a} .$$

Consider a four-scalar  $\bar{\phi}(\eta, \mathbf{x}) = \phi(\eta) + \delta\phi(\eta, \mathbf{x})$ . Under the transformation laws Eq. (A1), the perturbed part behaves as  $\delta\phi \mapsto \delta\phi - \phi'\zeta^0$ . Hence, one may also construct field-dependent gauge-invariant variables such as

$$-\mathcal{R} = \varphi_{\delta\phi} \equiv \varphi - \frac{\mathcal{H}}{\phi'}\delta\phi \equiv -\frac{\mathcal{H}}{\phi'}\delta\phi_\varphi , \quad \text{or} \quad \delta\phi_\chi \equiv \delta\phi - \frac{\phi'}{a}\chi \equiv -\frac{\phi'}{a}\chi_{\delta\phi} .$$

Choosing the uniform-field gauge  $\delta\phi \equiv 0$  as the temporal gauge condition promotes the perturbed variable  $\varphi$  to become a gauge-invariant quantity, which we denote by  $\varphi_{\delta\phi}$ . Similarly,  $-\frac{\mathcal{H}}{\phi'}\delta\phi_\varphi$  becomes gauge-invariant in the

uniform-curvature gauge,  $\varphi \equiv 0$ . Up to a sign, it is equal to the curvature perturbation in the comoving gauge  $\mathcal{R}$  [37]. The longitudinal gauge ( $\beta = \gamma = 0$ , hence  $\chi = 0$ ) renders the field perturbation  $\delta\phi$  gauge-invariant, which we denote by  $\delta\phi_\chi$ . We then find

$$\delta\phi_\chi = \frac{\phi'}{\mathcal{H}} [\Phi - \varphi_{\delta\phi}] = \frac{\phi'}{\mathcal{H}} [\Phi + \mathcal{R}] .$$

If we are to consider the longitudinal gauge to fix the temporal degree of freedom, the Bardeen potentials then coincide with the perturbations in the metric,  $\Phi = \varphi$  and  $\Psi = \tau$ , while  $\delta\phi_\chi = \delta\phi$ .

## 2. Background evolution

If we neglect perturbations in the metric and field, the infinitesimal line element reduces to  $ds^2 = a^2(\eta)(-d\eta^2 + d\vec{x}^2)$ . A solution for the background evolution is then required to satisfy the Einstein equation and the dynamical equation for the scalar field,

$$G_\nu^\mu = \frac{1}{2\mathcal{F}} \left[ T^{(0)\mu}_\nu + \alpha' \lambda T^{(c)\mu}_\nu \right] , \quad (\text{A3})$$

$$\phi'' + 2\mathcal{H}\phi' - \frac{1}{2\omega} \left[ 6\mathcal{F}_{,\phi} (\mathcal{H}' + \mathcal{H}^2) - \omega_{,\phi} \phi'^2 - 2a^2 \mathcal{V}_{,\phi} + \alpha' \lambda a^2 \Delta_\phi^{(c)} \right] = 0 , \quad (\text{A4})$$

where  $_{,\phi} \equiv d/d\phi$ , and the different contributions are

$$\begin{aligned} G_0^0 &= -3a^{-2}\mathcal{H}^2 , \\ G_j^i &= -a^{-2}(2\mathcal{H}' + \mathcal{H}^2)\delta_j^i , \\ T^{(0)0}_0 &= a^{-2} \left[ 6\mathcal{F}_{,\phi} \mathcal{H}\phi' - \omega\phi'^2 - 2a^2 \mathcal{V} \right] , \\ T^{(0)i}_j &= a^{-2} \left[ 2\mathcal{F}_{,\phi} (\phi'' + \mathcal{H}\phi') + (\omega + 2\mathcal{F}_{,\phi\phi}) \phi'^2 - 2a^2 \mathcal{V} \right] \delta_j^i , \\ T^{(c_1)0}_0 &= 24c_1 a^{-4} \xi_{,\phi} \phi' \mathcal{H}^3 , \\ T^{(c_2)0}_0 &= -9c_2 a^{-4} \xi \phi'^2 \mathcal{H}^2 , \\ T^{(c_3)0}_0 &= c_3 a^{-4} \phi'^3 (\xi_{,\phi} \phi' - 6\xi \mathcal{H}) , \\ T^{(c_4)0}_0 &= -3c_4 a^{-4} \xi \phi'^4 , \\ T^{(c_1)i}_j &= 8c_1 a^{-4} \left[ (\xi_{,\phi\phi} \phi'^2 + \xi_{,\phi} \phi'') \mathcal{H}^2 + \xi_{,\phi} \phi' \mathcal{H} (2\mathcal{H}' - \mathcal{H}^2) \right] \delta_j^i , \\ T^{(c_2)i}_j &= -c_2 a^{-4} \phi' \left[ \xi \phi' (2\mathcal{H}' + \mathcal{H}^2) + 4\xi (\phi'' - \phi' \mathcal{H}) \mathcal{H} + 2\xi_{,\phi} \phi'^2 \mathcal{H} \right] \delta_j^i , \\ T^{(c_3)i}_j &= -c_3 a^{-4} \phi'^2 \left[ 2\xi (\phi'' - \phi' \mathcal{H}) + \xi_{,\phi} \phi'^2 \right] \delta_j^i , \\ T^{(c_4)i}_j &= c_4 a^{-4} \xi \phi'^4 \delta_j^i , \\ \Delta_\phi^{(c_1)} &= 24c_1 a^{-4} \xi_{,\phi} \mathcal{H}' \mathcal{H}^2 , \\ \Delta_\phi^{(c_2)} &= -3c_2 a^{-4} \mathcal{H} \left[ \xi_{,\phi} \phi'^2 \mathcal{H} + 2\xi (\phi'' \mathcal{H} + 2\phi' \mathcal{H}') \right] , \\ \Delta_\phi^{(c_3)} &= c_3 a^{-4} \phi' \left[ \xi_{,\phi\phi} \phi'^3 + 4\xi_{,\phi} \phi' (\phi'' - \mathcal{H}\phi') - 6\xi (\phi' \mathcal{H}' + 2\phi'' \mathcal{H}) \right] , \\ \Delta_\phi^{(c_4)} &= -3c_4 a^{-4} \phi'^2 \left[ \xi_{,\phi} \phi'^2 + 4\xi \phi'' \right] . \end{aligned}$$

### 3. Linear perturbations

Assuming that there exists a class of background cosmologies which satisfy Eqs. (A3)–(A4) at all times, the equations of motion for small perturbations linearised about the background metric are

$$\delta G_\nu^\mu = \frac{1}{2\mathcal{F}} \left[ \delta T^{(0)\mu}_\nu + \alpha' \lambda \delta T^{(c)\mu}_\nu - 2\mathcal{F}_{,\phi} G_\nu^\mu \delta \phi \right]. \quad (\text{A5})$$

Following [35], the purely geometrical part obeys

$$\begin{aligned} \delta G_0^0 &= (\delta G_0^0)^{(GI)} + (G_0^0)' \frac{\chi}{a}, \\ \delta G_j^0 &= (\delta G_j^0)^{(GI)} + \left( G_0^0 - \frac{1}{3} G_k^k \right) \left( \frac{\chi}{a} \right)_{|j}, \\ \delta G_j^i &= (\delta G_j^i)^{(GI)} + (G_j^i)' \frac{\chi}{a}, \end{aligned}$$

and so do the terms on the right hand side of Eq. (A5). Using the background solution, the gauge-invariant equations of motion for small perturbations linearised about the background metric and field are

$$\begin{aligned} (\delta G_0^0)^{(GI)} &= \frac{1}{2\mathcal{F}} \left[ \left( \delta T^{(0)0}_0 \right)^{(GI)} + \alpha' \lambda \left( \delta T^{(c)0}_0 \right)^{(GI)} - 2\mathcal{F}_{,\phi} G_0^0 \delta \phi \chi \right], \\ (\delta G_j^0)^{(GI)} &= \frac{1}{2\mathcal{F}} \left[ \left( \delta T^{(0)0}_j \right)^{(GI)} + \alpha' \lambda \left( \delta T^{(c)0}_j \right)^{(GI)} \right], \\ (\delta G_j^i)^{(GI)} &= \frac{1}{2\mathcal{F}} \left[ \left( \delta T^{(0)i}_j \right)^{(GI)} + \alpha' \lambda \left( \delta T^{(c)i}_j \right)^{(GI)} - 2\mathcal{F}_{,\phi} G_j^i \delta \phi \chi \right], \\ (\delta G)^{(GI)}_{tf} &= \frac{1}{2\mathcal{F}} \left[ \left( \delta T^{(0)} \right)^{(GI)}_{tf} + \alpha' \lambda \left( \delta T^{(c)} \right)^{(GI)}_{tf} \right]. \end{aligned}$$

Here, we defined  $(T)_{tf} \equiv T_j^i - \frac{1}{3} T_k^k \delta_j^i$ , i.e., the tracefree part of a tensorial quantity  $T_j^i$ . Then, using gauge-invariant quantities, the dynamical equation for the scalar field  $\phi$  reads

$$\begin{aligned} 0 &= 2\omega \delta \phi''_\chi + 2(2\mathcal{H}\omega + \omega_{,\phi} \phi') \delta \phi'_\chi \\ &\quad + \left[ 2\omega_{,\phi}(\phi'' + 2\mathcal{H}\phi') - 6\mathcal{F}_{,\phi\phi}(\mathcal{H}' + \mathcal{H}^2) + \omega_{,\phi\phi}\phi'^2 + 2a^2\mathcal{V}_{,\phi\phi} - 2\omega\Delta \right] \delta \phi_\chi \\ &\quad - 6\mathcal{F}_{,\phi}\Phi'' - 2(3\mathcal{F}_{,\phi}\mathcal{H} - \omega\phi')(3\Phi' - \Psi') + 4a^2\mathcal{V}_{,\phi}\Psi + 2\mathcal{F}_{,\phi}\Delta(2\Phi + \Psi) \\ &\quad - \alpha' \lambda a^2 \left[ 2\Delta_\phi^{(c)} \Psi + \left( \delta \Delta_\phi^{(c)} \right)^{(GI)} \right], \end{aligned}$$

where we used

$$\delta \Delta_\phi^{(c_i)} = \left( \delta \Delta_\phi^{(c_i)} \right)^{(GI)} + \left( \Delta_\phi^{(c_i)} \right)' \frac{\chi}{a}.$$

The gauge-invariant components entering the above equations are

$$\begin{aligned} (\delta G_0^0)^{(GI)} &= -2a^{-2} [3\mathcal{H}(\Phi' - \Psi\mathcal{H}) - \Delta\Phi], \\ (\delta G_j^0)^{(GI)} &= 2a^{-2} [\Phi' - \Psi\mathcal{H}]_{|j}, \\ (\delta G_j^i)^{(GI)} &= -2a^{-2} [\Phi'' + (2\Phi' - \Psi')\mathcal{H} - \Psi(2\mathcal{H}' + \mathcal{H}^2)] \delta_j^i \\ &\quad - a^{-2} (\nabla^i \nabla_j - \Delta \delta_j^i)(\Phi + \Psi), \\ \left( \delta T^{(0)0}_0 \right)^{(GI)} &= a^{-2} \left[ 6\mathcal{F}_{,\phi\phi}\mathcal{H}\phi' - \omega_{,\phi}\phi'^2 - 2a^2\mathcal{V}_{,\phi} - 2\mathcal{F}_{,\phi}\Delta \right] \delta \phi_\chi \\ &\quad + 2a^{-2} \left[ (3\mathcal{H}\mathcal{F}_{,\phi} - \omega\phi')\delta \phi'_\chi + 3\mathcal{F}_{,\phi}\phi'\Phi' - \phi'(6\mathcal{F}_{,\phi}\mathcal{H} - \omega\phi')\Psi \right], \\ \left( \delta T^{(0)0}_j \right)^{(GI)} &= -2a^{-2} \left[ \mathcal{F}_{,\phi}\delta \phi'_\chi + (\mathcal{F}_{,\phi\phi}\phi' + \omega\phi' - \mathcal{F}_{,\phi}\mathcal{H})\delta \phi_\chi - \mathcal{F}_{,\phi}\phi'\Psi \right]_{|j}, \end{aligned}$$

$$\begin{aligned}
\left(\delta T^{(0)}_j^i\right)^{(GI)} &= a^{-2} \left\{ 2\mathcal{F}_{,\phi}\delta\phi''_\chi + 2\left[\mathcal{F}_{,\phi}\mathcal{H} + \phi'(\omega + 2\mathcal{F}_{,\phi\phi})\right]\delta\phi'_\chi \right. \\
&\quad + \left[2\mathcal{F}_{,\phi\phi}(\phi'' + \mathcal{H}\phi') + (\omega_{,\phi} + 2\mathcal{F}_{,\phi\phi\phi})\phi'^2 - \frac{2}{a^2}\mathcal{V}_{,\phi}\right]\delta\phi_\chi \\
&\quad \left. + 2\mathcal{F}_{,\phi}\phi'(2\Phi' - \Psi') - 2\left[2\mathcal{F}_{,\phi}(\phi'' + \mathcal{H}\phi') + \phi'^2(\omega + 2\mathcal{F}_{,\phi\phi})\right]\Psi \right\} \delta_j^i \\
&\quad + 2a^{-2}\mathcal{F}_{,\phi}(\nabla^i\nabla_j - \Delta\delta_j^i)\delta\phi_\chi,
\end{aligned}$$

$$\begin{aligned}
\left(\delta T^{(c_1)}_0^0\right)^{(GI)} &= 8c_1a^{-4}\mathcal{H}\left[3\xi_{,\phi}\mathcal{H}^2\delta\phi'_\chi + 3\xi_{,\phi\phi}\mathcal{H}^2\phi'\delta\phi_\chi + 3\xi_{,\phi}\mathcal{H}\phi'(3\Phi' - 4\mathcal{H}\Psi) \right. \\
&\quad \left. - \xi_{,\phi}\Delta(2\phi'\Phi + \mathcal{H}\delta\phi_\chi) \right], \\
\left(\delta T^{(c_1)}_j^0\right)^{(GI)} &= -8c_1a^{-4}\mathcal{H}\left[\xi_{,\phi}\mathcal{H}\delta\phi'_\chi + \mathcal{H}(\xi_{,\phi\phi}\phi' - \xi_{,\phi}\mathcal{H})\delta\phi_\chi + \xi_{,\phi}\phi'(2\Phi' - 3\mathcal{H}\Psi)\right]_{|j}, \\
\left(\delta T^{(c_1)}_j^i\right)^{(GI)} &= 8c_1a^{-4} \left\{ \xi_{,\phi}\mathcal{H}^2\delta\phi''_\chi + \mathcal{H}(2\xi_{,\phi\phi}\mathcal{H}\phi' + 2\xi_{,\phi}\mathcal{H}' - \xi_{,\phi}\mathcal{H}^2)\delta\phi'_\chi \right. \\
&\quad + \mathcal{H}(\xi_{,\phi\phi\phi}\mathcal{H}\phi'^2 + \xi_{,\phi\phi}\{2\mathcal{H}'\phi' + \mathcal{H}\phi'' - \mathcal{H}^2\phi'\})\delta\phi_\chi \\
&\quad + 2\xi_{,\phi}\mathcal{H}\phi'\Phi'' + 2(\xi_{,\phi\phi}\mathcal{H}\phi'^2 + \xi_{,\phi}\mathcal{H}'\phi' + \xi_{,\phi}\mathcal{H}\phi'')\Phi' - 3\xi_{,\phi}\mathcal{H}^2\phi'\Psi' \\
&\quad \left. - 4\mathcal{H}(\xi_{,\phi\phi}\mathcal{H}\phi'^2 + \xi_{,\phi}\{2\mathcal{H}'\phi' + \mathcal{H}\phi'' - \mathcal{H}^2\phi'\})\Psi \right\} \delta_j^i \\
&\quad + 8c_1a^{-4}(\nabla^i\nabla_j - \Delta\delta_j^i)\left[\xi_{,\phi}\mathcal{H}'\delta\phi_\chi + (\xi_{,\phi\phi}\phi'^2 + \xi_{,\phi}[\phi'' - \mathcal{H}\phi'])\Phi + \xi_{,\phi}\mathcal{H}\phi'\Psi\right],
\end{aligned}$$

$$\begin{aligned}
\left(\delta T^{(c_2)}_0^0\right)^{(GI)} &= -c_2a^{-4}\phi'\left[18\xi\mathcal{H}^2\delta\phi'_\chi + 9\xi_{,\phi}\mathcal{H}^2\phi'\delta\phi_\chi - 4\xi\mathcal{H}\Delta\delta\phi_\chi \right. \\
&\quad \left. + 18\xi\mathcal{H}\phi'(\Phi' - 2\mathcal{H}\Psi) - 2\xi\phi'\Delta\Phi\right], \\
\left(\delta T^{(c_2)}_j^0\right)^{(GI)} &= 2c_2a^{-4}\phi'\left[2\xi\mathcal{H}\delta\phi'_\chi + \mathcal{H}(\xi_{,\phi}\phi' - 3\xi\mathcal{H})\delta\phi_\chi + \xi\phi'(\Phi' - 3\mathcal{H}\Psi)\right]_{|j}, \\
\left(\delta T^{(c_2)}_j^i\right)^{(GI)} &= -c_2a^{-4} \left\{ 4\xi\mathcal{H}\phi'\delta\phi''_\chi + 2\left[\xi(2\mathcal{H}'\phi' + 2\mathcal{H}\phi'' - 3\mathcal{H}^2\phi') + 3\xi_{,\phi}\mathcal{H}\phi'^2\right]\delta\phi'_\chi \right. \\
&\quad + \phi'\left[2\xi_{,\phi\phi}\mathcal{H}\phi'^2 + \xi_{,\phi}(2\mathcal{H}'\phi' + 4\mathcal{H}\phi'' - 3\mathcal{H}^2\phi')\right]\delta\phi_\chi \\
&\quad + 2\xi\phi'^2\Phi'' + 2\phi'(\xi_{,\phi}\phi'^2 + 2\xi\phi'')\Phi' - 6\xi\mathcal{H}\phi'^2\Psi' \\
&\quad \left. - 4\phi'\left[2\xi_{,\phi}\mathcal{H}\phi'^2 + \xi(2\mathcal{H}'\phi' + 4\mathcal{H}\phi'' - 3\mathcal{H}^2\phi')\right]\Psi \right\} \delta_j^i \\
&\quad - c_2a^{-4}(\nabla^i\nabla_j - \Delta\delta_j^i)\left[(\xi_{,\phi}\phi'^2 + 2\xi\phi'')\delta\phi_\chi - \xi\phi'^2(\Phi - \Psi)\right],
\end{aligned}$$

$$\begin{aligned}
\left(\delta T^{(c_3)}_0^0\right)^{(GI)} &= c_3a^{-4}\phi'^2 \left\{ 2(2\xi_{,\phi}\phi' - 9\xi\mathcal{H})\delta\phi'_\chi + \phi'(\xi_{,\phi\phi}\phi' - 6\xi_{,\phi}\mathcal{H})\delta\phi_\chi + 2\xi\Delta\delta\phi_\chi \right. \\
&\quad \left. - 6\xi\phi'\Phi' - 4\phi'(\xi_{,\phi}\phi' - 6\xi\mathcal{H})\Psi \right\}, \\
\left(\delta T^{(c_3)}_j^0\right)^{(GI)} &= 2c_3a^{-4}\phi'\left[\xi\phi'\delta\phi'_\chi + \phi'(\xi_{,\phi}\phi' - 3\xi\mathcal{H})\delta\phi_\chi - \xi\phi'^2\Psi\right]_{|j}, \\
\left(\delta T^{(c_3)}_j^i\right)^{(GI)} &= -c_3a^{-4}\phi' \left\{ 2\xi\phi'\delta\phi''_\chi + 2[2\xi_{,\phi}\phi'^2 + \xi(2\phi'' - 3\mathcal{H}\phi')]\delta\phi'_\chi - 2\xi\phi'^2\Psi' \right. \\
&\quad + \phi'\left[\xi_{,\phi\phi}\phi'^2 + 2\xi_{,\phi}(\phi'' - \mathcal{H}\phi')\right]\delta\phi_\chi \\
&\quad \left. - 4\phi'\left[\xi_{,\phi}\phi'^2 + 2\xi(\phi'' - \mathcal{H}\phi')\right]\Psi \right\} \delta_j^i,
\end{aligned}$$

$$\begin{aligned}
\left(\delta T^{(c_4)}_0\right)^{(GI)} &= -3c_4 a^{-4} \phi'^3 [4\xi\delta\phi'_\chi + \xi_{,\phi}\phi'\delta\phi_\chi - 4\xi\phi'\Psi] , \\
\left(\delta T^{(c_4)}_j\right)^{(GI)} &= -4c_4 a^{-4} \xi\phi'^3 \delta\phi_{\chi|j} , \\
\left(\delta T^{(c_4)}_j\right)^{(GI)} &= c_4 a^{-4} \phi'^3 [4\xi\delta\phi'_\chi + \xi_{,\phi}\phi'\delta\phi_\chi - 4\xi\phi'\Psi] \delta_j^i , \\
\left(\delta\Delta_\phi^{(c_1)}\right)^{(GI)} &= 8c_1 a^{-4} \xi_{,\phi} [3\mathcal{H}^2\Phi'' + 3\mathcal{H}(\mathcal{H}^2 + 2\mathcal{H}')\Phi' - 3\mathcal{H}^2(\mathcal{H}\Psi' + 4\mathcal{H}'\Psi) \\
&\quad - \Delta(2\mathcal{H}'\Phi + \mathcal{H}^2\Psi)] + 24c_1 a^{-4} \xi_{,\phi\phi} \mathcal{H}^2 \mathcal{H}' \delta\phi_\chi , \\
\left(\delta\Delta_\phi^{(c_2)}\right)^{(GI)} &= -c_2 a^{-4} [6\xi\mathcal{H}^2\delta\phi''_\chi + 6\mathcal{H}(\xi_{,\phi}\phi'\mathcal{H} + 2\xi\mathcal{H}')\delta\phi'_\chi - 2\xi(2\mathcal{H}' + \mathcal{H}^2)\Delta\delta\phi_\chi \\
&\quad + 3\mathcal{H}\{\xi_{,\phi\phi}\mathcal{H}\phi'^2 + 2\xi_{,\phi}(\mathcal{H}\phi'' + 2\mathcal{H}'\phi')\}\delta\phi_\chi + 12\xi\mathcal{H}\phi'\Phi'' \\
&\quad + 6\{\xi_{,\phi}\mathcal{H}\phi'^2 + \xi(2\mathcal{H}\phi'' + 2\mathcal{H}'\phi' + 3\mathcal{H}^2\phi')\}\Phi' \\
&\quad - 2(\xi_{,\phi}\phi'^2 + 2\xi\phi'')\Delta\Phi - 18\xi\mathcal{H}^2\phi'\Psi' \\
&\quad - 12\mathcal{H}\{\xi_{,\phi}\mathcal{H}\phi'^2 + 2\xi(2\mathcal{H}'\phi' + \mathcal{H}\phi'')\}\Psi - 4\xi\mathcal{H}\phi'\Delta\Psi] , \\
\left(\delta\Delta_\phi^{(c_3)}\right)^{(GI)} &= c_3 a^{-4} [4\phi'(\xi_{,\phi}\phi' - 3\xi\mathcal{H})\delta\phi''_\chi - 2\phi'^2(2\xi_{,\phi}\phi' - 9\xi\mathcal{H})\Psi' + 2\xi\phi'^2\Delta\Psi \\
&\quad + 4\{\xi_{,\phi\phi}\phi'^3 + \xi_{,\phi}\phi'(2\phi'' - 3\mathcal{H}\phi') - 3\xi(\mathcal{H}'\phi' + \mathcal{H}\phi'')\}\delta\phi'_\chi \\
&\quad + \phi'\{\xi_{,\phi\phi\phi}\phi'^3 + 4\xi_{,\phi\phi}\phi'(\phi'' - \mathcal{H}\phi') - 6\xi_{,\phi}(2\mathcal{H}\phi'' + \mathcal{H}'\phi')\}\delta\phi_\chi \\
&\quad + 4\xi(\phi'' + \mathcal{H}\phi')\Delta\delta\phi_\chi - 6\xi\phi'^2\Phi'' - 6\xi\phi'(2\phi'' + 3\mathcal{H}\phi')\Phi' \\
&\quad - 4\phi'\{\xi_{,\phi\phi}\phi'^3 - 4\xi_{,\phi}\phi'(\mathcal{H}\phi' - \phi'') - 6\xi(\mathcal{H}'\phi' + 2\mathcal{H}\phi'')\}\Psi] , \\
\left(\delta\Delta_\phi^{(c_4)}\right)^{(GI)} &= -c_4 a^{-4} [12\xi\phi'^2\delta\phi''_\chi + 12\phi'(\xi_{,\phi}\phi'^2 + 2\xi\phi'')\delta\phi'_\chi + 3\phi'^2(\xi_{,\phi\phi}\phi'^2 + 4\xi_{,\phi}\phi'')\delta\phi_\chi \\
&\quad - 4\xi\phi'^2\Delta\delta\phi_\chi + 12\xi\phi'^3(\Phi' - \Psi') - 12\phi'^2(\xi_{,\phi}\phi'^2 + 4\xi\phi'')\Psi] .
\end{aligned}$$

#### 4. Structure of the perturbation equations

To visualise the structure of these complicated contributions, we write down the perturbation equations in matrix form,

$$\mathcal{M}^{(\Phi)}\hat{\Phi} + \mathcal{M}^{(\Psi)}\hat{\Psi} + \mathcal{M}^{(\delta\phi_\chi)}\hat{\delta\phi}_\chi = 0 , \quad (\text{A6})$$

where we use  $\hat{x} = (x'', x', \Delta x, x)^T$ ,  $x \in \{\Phi, \Psi, \delta\phi_\chi\}$ . According to the gauge-invariant contributions of Sec. A 3, the  $\mathcal{M}^{(x)}$  represent  $5 \times 4$  matrices, each row of Eq. (A6) yielding one of the five component perturbation equations. Here we choose that the first three rows correspond to the  $(0)$ ,  $(0)_j$  and  $(i)$  part of the perturbed Einstein equation, while the fourth row correspond to the  $(\phi)$  perturbation equations. Finally, the tracefree part of the Einstein equation yields the fifth row. Explicitly, after use of the background equations, the matrices entering Eq. (A6) have the following structure

$$\begin{aligned}
\mathcal{M}^{(\Phi)} &= \begin{pmatrix} 0 & -6E_1 & 2E_2 & 0 \\ 0 & 2E_2 & 0 & 0 \\ -6E_2 & -6E_9 & 2E_3 & 0 \\ -6E_4 & 6E_8 & 2E_6 & 0 \\ 0 & 0 & 0 & -E_3 \end{pmatrix} , & \mathcal{M}^{(\Psi)} &= \begin{pmatrix} 0 & 0 & 0 & E_{17} \\ 0 & 0 & 0 & -2E_1 \\ 0 & 6E_1 & 2E_2 & 3E_{18} \\ 0 & -2E_5 & 2E_4 & 2E_{19} \\ 0 & 0 & 0 & -E_2 \end{pmatrix} , \\
\mathcal{M}^{(\delta\phi_\chi)} &= \begin{pmatrix} 0 & 2E_5 & 2E_4 & -E_{10} \\ 0 & 2E_4 & 0 & 2E_{11} \\ -6E_4 & -6E_{20} & 2E_6 & -3E_{12} \\ 2E_{14} & 2E_{15} & -2E_{16} & E_{13} \\ 0 & 0 & 0 & -E_6 \end{pmatrix} ,
\end{aligned}$$

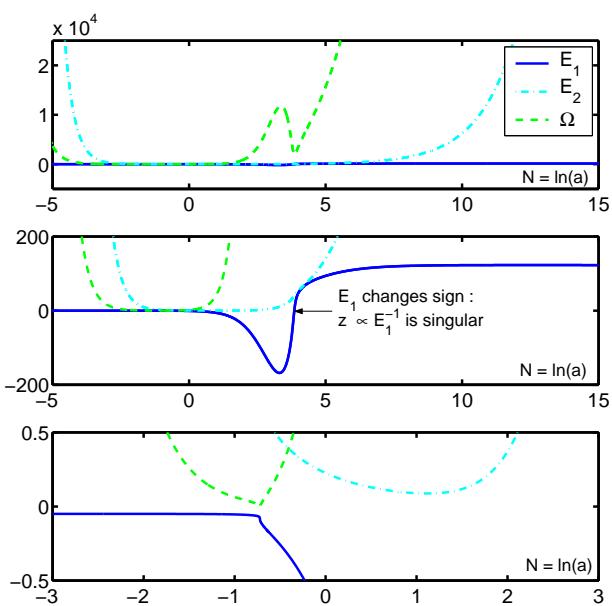


FIG. 3: Here we illustrate the evolution of the quantities  $E_1$ ,  $E_2$  and  $\Omega$  as functions of the number of e-folds,  $N = \ln(a)$ , in the context of the high-curvature transition of short duration used to determine the spectral distribution of Fig. 2. As explained in the main text,  $E_1$  has to change sign in a bouncing universe. The pump field  $z = (\phi' E_2/E_1^2)^{1/2}$  and  $z'/z$  are thus singular, which invalidates the use of the homogeneous second-order differential equation for the curvature perturbation in the comoving gauge,  $\mathcal{R}'' + 2\frac{z'}{z}\mathcal{R}' + sk^2\mathcal{R} = 0$ . Since  $E_2$  and  $\Omega$  do not change sign, the homogeneous systems of linear differential equations Eq. (19) and Eq. (25) remain well-defined at all times, and can be used to determine the spectral distribution of adiabatic density perturbations long after the bounce.

where we have highlighted the components which are null at all times. It is then obvious that the tracefree part of the Einstein equation, i.e., the fifth row of Eq. (A6), implies a linear relation among the perturbation variables

$$\Psi = c_\Phi \Phi + c_{\delta\phi_\chi} \delta\phi_\chi , \quad (\text{A7})$$

where the coefficients  $c_\Phi = -E_3 E_2^{-1}$  and  $c_{\delta\phi_\chi} = -E_6 E_2^{-1}$  are functions of the background evolution, but do not depend explicitly on the wavenumber  $k$ . Provided that the background evolution satisfies  $E_2 \neq 0$  at all times, we can remove  $\Psi$  from Eq. (A6), which yields a system of two second-order differential equations and two constraints  $\tilde{\mathcal{M}}^{(\Phi)}\hat{\Phi} + \tilde{\mathcal{M}}^{(\delta\phi_\chi)}\delta\hat{\phi}_\chi = 0$ , the  $\tilde{\mathcal{M}}^{(x)}$  corresponding to  $4 \times 4$  matrices. It turns out that some of these equations are redundant, and one can show that the first two rows, i.e., the  $(0)_0$  and  $(0)_j$  components of the perturbed Einstein equation are sufficient to fully characterise the dynamical evolution of linear perturbations. Explicitly, we get

$$\begin{pmatrix} -6E_1 & 2E_2 & c_\Phi E_{17} \\ E_2 & 0 & -c_\Phi E_1 \end{pmatrix} \begin{pmatrix} \Phi' \\ \Delta\Phi \\ \Phi \end{pmatrix} + \begin{pmatrix} 2E_5 & 2E_4 & -E_{10} + c_{\delta\phi_\chi} E_{17} \\ E_4 & 0 & E_{11} - c_{\delta\phi_\chi} E_1 \end{pmatrix} \begin{pmatrix} \delta\phi'_\chi \\ \Delta\delta\phi_\chi \\ \delta\phi_\chi \end{pmatrix} = 0 . \quad (\text{A8})$$

In Fourier space, these first-order equations can then be written as  $\mathcal{A}X' + \mathcal{B}X = 0$  where  $X \equiv (\Phi, \delta\phi_\chi)^T$  and

$$\mathcal{A}(\eta) = \begin{pmatrix} -6E_1 & 2E_5 \\ E_2 & E_4 \end{pmatrix} , \quad \mathcal{B}(\eta, k) = \begin{pmatrix} c_\Phi E_{17} - 2k^2 E_2 & c_{\delta\phi_\chi} E_{17} - E_{10} - 2k^2 E_4 \\ -c_\Phi E_1 & E_{11} - c_{\delta\phi_\chi} E_1 \end{pmatrix} . \quad (\text{A9})$$

Provided that the matrix  $\mathcal{A}$  is invertible, i.e.,  $\det(\mathcal{A}) = -2\Omega \neq 0$  at all times, we then have  $X' = \mathcal{C}(\eta, k)X$  with  $\mathcal{C} \equiv -\mathcal{A}^{-1}\mathcal{B}$ . Figure 3 shows explicitly that  $\Omega$  does not change sign during the evolution for the background evolution discussed in the main text. We can then proceed further by replacing the gauge-invariant field perturbation,  $\delta\phi_\chi = \frac{\phi'}{\mathcal{H}}(\Phi + \mathcal{R})$ , which yields a system of first-order coupled differential equations involving the Bardeen potential  $\Phi(\eta, k)$  and the curvature perturbation in the comoving gauge  $\mathcal{R}(\eta, k)$ .

To be complete, we finally present the explicit form of the coefficients of the matrices  $\mathcal{M}^{(\Phi)}$ ,  $\mathcal{M}^{(\Psi)}$  and  $\mathcal{M}^{(\delta\phi_\chi)}$ . Not all of these quantities are independent and one may use the background equations (including the higher-order corrections) for additional simplifications.

$$\begin{aligned}
E_1 &\equiv a^2 (\mathcal{F}' + 2\mathcal{FH} + aQ_g) , \\
E_2 &\equiv a^2 (2\mathcal{F} + Q_b) , \\
E_3 &\equiv a^2 (2\mathcal{F} + Q_i) , \\
E_4 &\equiv a^2 \phi'^{-1} (\mathcal{F}' + aQ_a) , \\
E_5 &\equiv a^2 \phi'^{-1} (\omega \phi'^2 - 3\mathcal{F}'\mathcal{H} + a^2 Q_n) , \\
E_6 &\equiv a^2 \phi'^{-1} (2\mathcal{F}' + aQ_j) , \\
E_7 &\equiv a^2 (2\mathcal{F} + Q_m) , \\
E_8 &\equiv a^2 \phi'^{-1} (\omega \phi'^2 - 3\mathcal{F}'\mathcal{H} + a^2 Q_l) , \\
E_9 &\equiv 2a^2 (\mathcal{F}' + 2\mathcal{H}\mathcal{F} + aQ_k) , \\
E_{10} &\equiv a^2 \phi'^{-2} (6\mathcal{H} [\mathcal{F}''\phi' - \mathcal{F}'\phi''] + 6\mathcal{F}'\mathcal{H}^2\phi' - \omega'\phi'^3 - 2a^2\mathcal{V}'\phi' + a^4Q_s) , \\
E_{11} &\equiv a^2 \phi'^{-2} (\mathcal{F}''\phi' - \mathcal{F}'\phi'' - \mathcal{F}'\mathcal{H}\phi' + \omega\phi'^3 + a^3Q_t) , \\
E_{12} &\equiv a^2 \phi'^{-3} (2\mathcal{F}'''\phi'^2 + 2\mathcal{F}'' [\mathcal{H}\phi' - 2\phi'']\phi' + \omega'\phi'^4 - 2a^2\mathcal{V}'\phi'^2 \\
&\quad + 2\mathcal{F}' [\mathcal{H}^2\phi'^2 + 2\mathcal{H}'\phi'^2 - \phi'\phi''' + 2\phi''^2 - \mathcal{H}\phi'\phi''] + a^5Q_u) , \\
E_{13} &\equiv a^2 \phi'^{-3} (-6 [\mathcal{F}''\phi' - \mathcal{F}'\phi''] [\mathcal{H}' + \mathcal{H}^2] + \omega''\phi'^3 + \omega'\phi'^2 [4\mathcal{H}\phi' + \phi''] \\
&\quad + 2a^2\mathcal{V}''\phi' - 2a^2\mathcal{V}'\phi'' + a^5Q_v) , \\
E_{14} &\equiv a^2 \phi'^{-2} (\omega\phi'^2 + a^2Q_c) , \\
E_{15} &\equiv a^2 (\omega' + 2\mathcal{H}\omega + aQ_w) , \\
E_{16} &\equiv a^2 \phi'^{-2} (\omega\phi'^2 + a^2Q_o) , \\
E_{17} &\equiv 2a^2 (6\mathcal{F}'\mathcal{H} + 6\mathcal{F}\mathcal{H}^2 - \omega\phi'^2 + a^2Q_p) , \\
E_{18} &\equiv 2a^2 [2\mathcal{F}'' + 2\mathcal{F}'\mathcal{H} + 2\mathcal{F} (2\mathcal{H}' + \mathcal{H}^2) + \omega\phi'^2 + a^2Q_q] , \\
E_{19} &\equiv a^4 \phi'^{-1} (2V' + aQ_r) , \\
E_{20} &\equiv a^2 \phi'^{-2} [2 (\mathcal{F}''\phi' - \mathcal{F}''\phi'') + \mathcal{F}'\mathcal{H}\phi' + \omega\phi'^3 + a^3Q_x] ,
\end{aligned}$$

$$\begin{aligned}
Q_a &\equiv \alpha'\lambda a^{-3} [4c_1\xi'\mathcal{H}^2 - 2c_2\xi\mathcal{H}\phi'^2 - c_3\xi\phi'^3] , \\
Q_b &\equiv \alpha'\lambda a^{-2} [8c_1\xi'\mathcal{H} - c_2\xi\phi'^2] , \\
Q_c &\equiv \alpha'\lambda a^{-4} \phi'^2 [3c_2\xi\mathcal{H}^2 + 2c_3\phi' (3\xi\mathcal{H} - \xi') + 6c_4\xi\phi'^2] , \\
Q_d &\equiv 2\alpha'\lambda a^{-4} \phi'^2 [c_2\xi (\mathcal{H}' - \mathcal{H}^2) + c_3 (\xi'\phi' + \xi\phi'' - 2\xi\mathcal{H}\phi') - 2c_4\xi\phi'^2] , \\
Q_e &\equiv 2\alpha'\lambda a^{-3} [8c_1\xi' (\mathcal{H}' - \mathcal{H}^2) - c_2\phi' (\xi'\phi' + 2\xi\phi'' - 4\xi\phi'\mathcal{H}) + 2c_3\xi\phi'^3] , \\
Q_f &\equiv -2\alpha'\lambda a^{-2} [4c_1 (\xi'' - 2\xi'\mathcal{H}) + c_2\xi\phi'^2] , \\
Q_g &\equiv \alpha'\lambda a^{-3} [12c_1\xi'\mathcal{H}^2 - 3c_2\xi\mathcal{H}\phi'^2 - c_3\xi\phi'^3] , \\
Q_h &\equiv 4\alpha'\lambda a^{-2} c_1\xi'' , \\
Q_i &\equiv \alpha'\lambda a^{-2} [8c_1 (\xi'' - \xi'\mathcal{H}) + c_2\xi\phi'^2] , \\
Q_j &\equiv \alpha'\lambda a^{-3} [8c_1\xi'\mathcal{H}' - c_2\phi' (\xi'\phi' + 2\xi\phi'')] , \\
Q_k &\equiv \frac{1}{2}\alpha'\lambda a^{-3} [8c_1 (\xi''\mathcal{H} + \xi'\mathcal{H}') - c_2\phi' (\xi'\phi' + 2\xi\phi'')] , \\
Q_l &\equiv \alpha'\lambda a^{-4} [-4c_1\xi'\mathcal{H} (2\mathcal{H}' + \mathcal{H}^2) + c_2\phi' (\xi'\mathcal{H}\phi' + \xi \{2\mathcal{H}'\phi' + 2\mathcal{H}\phi'' + 3\mathcal{H}^2\phi'\}) \\
&\quad + c_3\xi\phi'^2 (2\phi'' + 3\mathcal{H}\phi') + 2c_4\xi\phi'^4] , \\
Q_m &\equiv 8\alpha'\lambda c_1 a^{-2} \xi'\mathcal{H} , \\
Q_n &\equiv \alpha'\lambda a^{-4} [-12c_1\xi'\mathcal{H}^3 + 9c_2\xi\mathcal{H}^2\phi'^2 - c_3\phi'^3 (2\xi' - 9\xi\mathcal{H}) + 6c_4\xi\phi'^4] , \\
Q_o &\equiv \alpha'\lambda a^{-4} \phi'^2 [c_2\xi (2\mathcal{H}' + \mathcal{H}^2) + 2c_3\xi (\phi'' + \mathcal{H}\phi') + 2c_4\xi\phi'^2] , \\
Q_p &\equiv 2\alpha'\lambda a^{-4} [24c_1\xi'\mathcal{H}^3 - 9c_2\xi\mathcal{H}^2\phi'^2 + c_3\phi'^3 (\xi' - 6\xi\mathcal{H}) - 3c_4\xi\phi'^4] , \\
Q_q &\equiv 2\alpha'\lambda a^{-4} [8c_1\mathcal{H} (\xi''\mathcal{H} + 2\xi'\mathcal{H}' - \xi'\mathcal{H}^2) - c_2\phi' (2\xi'\mathcal{H}\phi' + 2\xi\mathcal{H}'\phi' + 4\xi\mathcal{H}\phi'' - 3\xi\mathcal{H}^2\phi')]
\end{aligned}$$

$$\begin{aligned}
& -c_3\phi'^2(\xi'\phi' + 2\xi\phi'' - 2\xi\mathcal{H}\phi') + c_4\xi\phi'^4 \Big] , \\
Q_r & \equiv \alpha'\lambda a^{-5} \left[ 24c_1\xi'\mathcal{H}'\mathcal{H}^2 - 3c_2\mathcal{H}\phi'(\xi'\mathcal{H}\phi' + 2\xi\mathcal{H}\phi'' + 4\xi\mathcal{H}'\phi') \right. \\
& \quad \left. + c_3\phi'^2(\xi''\phi' + \xi'\{3\phi'' - 4\mathcal{H}\phi'\} - 6\xi\{\mathcal{H}'\phi' + 2\mathcal{H}\phi''\}) - 3c_4\phi'^3(\xi'\phi' + 4\xi\phi'') \right] , \\
Q_s & \equiv \alpha'\lambda a^{-6} \left[ 24c_1(\xi''\phi' - \xi'\phi'')\mathcal{H}^3 - 9c_2\xi'\mathcal{H}^2\phi'^3 + c_3\phi'^3(\xi''\phi' - \xi'\phi'' - 6\xi'\mathcal{H}\phi') - 3c_4\xi'\phi'^5 \right] , \\
Q_t & \equiv \alpha'\lambda a^{-5} \left[ 4c_1\mathcal{H}^2(\xi''\phi' - \xi'\phi'' - \xi'\mathcal{H}\phi') - c_2\mathcal{H}\phi'^3(\xi' - 3\xi\mathcal{H}) - c_3\phi'^4(\xi' - 3\xi\mathcal{H}) + 2c_4\xi\phi'^5 \right] , \\
Q_u & \equiv \alpha'\lambda a^{-7} \left[ 8c_1\mathcal{H}(\xi'''\mathcal{H}\phi'^2 + \xi''\phi'\{2\mathcal{H}'\phi' - \mathcal{H}^2\phi' - 2\mathcal{H}\phi''\} - \xi'\{\mathcal{H}\phi'\phi''' + 2\mathcal{H}'\phi'\phi'' - 2\mathcal{H}\phi''^2 - \mathcal{H}^2\phi'\phi''\}) \right. \\
& \quad \left. - c_2\phi'^3(2\xi''\mathcal{H}\phi' + \xi'\{2\mathcal{H}'\phi' + 2\mathcal{H}\phi'' - 3\mathcal{H}^2\phi'\}) - c_3\phi'^4(\xi''\phi' + \xi'\phi'' - 2\xi'\mathcal{H}\phi') + c_4\xi'\phi'^6 \right] , \\
Q_v & \equiv \alpha'\lambda a^{-7} \left[ -24c_1\mathcal{H}'\mathcal{H}^2(\xi''\phi' - \xi'\phi'') + 3c_2\mathcal{H}\phi'^2(\xi''\mathcal{H}\phi' + \xi'\mathcal{H}\phi'' + 4\xi'\mathcal{H}'\phi') \right. \\
& \quad \left. + c_3\phi'^2(-\xi'''\phi'^2 + \xi''\phi'\{4\mathcal{H}\phi' - \phi''\} + \xi'\{8\mathcal{H}\phi'\phi'' + \phi'''^2 + \phi'\phi''' + 6\mathcal{H}'\phi'^2\}) \right. \\
& \quad \left. + 3c_4\phi'^4(\xi''\phi' + 3\xi'\phi'') \right] , \\
Q_w & \equiv \alpha'\lambda a^{-3} \left[ 3c_2\mathcal{H}(\xi'\mathcal{H} + 2\xi\mathcal{H}') + 6c_4\phi'(\xi'\phi' + 2\xi\phi'') \right. \\
& \quad \left. + 2c_3(3\xi\mathcal{H}'\phi' - \xi'\phi'' - \xi''\phi' + 3\xi\mathcal{H}\phi'' + 3\xi'\mathcal{H}\phi') \right] , \\
Q_x & \equiv \alpha'\lambda a^{-5} \left[ 4c_1\mathcal{H}(2\xi''\mathcal{H}\phi' - 2\xi'\mathcal{H}\phi'' + 2\xi'\mathcal{H}'\phi' - \xi'\mathcal{H}^2\phi') + c_3\phi'^3(-2\xi'\phi' + 3\xi\mathcal{H}\phi' - 2\xi\phi'') \right. \\
& \quad \left. - c_2\phi'^2(3\xi'\mathcal{H}\phi' + 2\xi\mathcal{H}\phi'' + 2\xi\mathcal{H}'\phi' - 3\xi\mathcal{H}^2\phi') + 2c_4\xi\phi'^5 \right] .
\end{aligned}$$